

QUANTITATIVE VOLUME SPACE FROM RIGIDITY WITH LOWER RICCI CURVATURE BOUND II

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ABSTRACT. This is the second paper of two in a series under the same title ([CRX]); both study the quantitative volume space form rigidity conjecture: a closed n -manifold of Ricci curvature at least $(n-1)H$, $H = \pm 1$ or 0 is diffeomorphic to a H -space form if for every ball of definite size on M , the lifting ball on the Riemannian universal covering space of the ball achieves an almost maximal volume, provided the diameter of M is bounded for $H \neq 1$.

In [CRX], we verified the conjecture for the case that M or its Riemannian universal covering space \tilde{M} is not collapsed for $H = 1$ or $H \neq 1$ respectively. In the present paper, we will verify this conjecture for the case that Ricci curvature is also bounded above, while the above non-collapsing condition is not required.

0. INTRODUCTION

This is the second paper of two in a series under the same title, concerning the quantitative version of the following volume space form rigidity.

Let M be a compact n -manifold of Ricci curvature bounded below by $(n-1)H$, a constant. For $p \in M$ and $r > 0$, the volume of the r -ball at p , $\text{vol}(B_r(p)) \leq \text{vol}(\underline{B}_r^H)$, and “=” if and only if the open ball $B_r(p)$ is isometric to \underline{B}_r^H (Bishop volume comparison), which denotes the r -ball in the n -dimensional simply connected H -space form.

The following statement is a consequence of the Bishop volume comparison.

Theorem 0.1. (*Volume space form rigidity*) *Let $\rho > 0$. If a compact n -manifold M satisfies*

$$\text{Ric}_M \geq (n-1)H, \quad \frac{\text{vol}(B_\rho(x^*))}{\text{vol}(\underline{B}_\rho^H)} = 1, \quad \forall x \in M,$$

then M is isometric to a space form of constant curvature H , where $\pi^ : (\widetilde{B_\rho(x)}, x^*) \rightarrow (B_\rho(x), x)$ is the (incomplete) Riemannian universal covering space.*

All H -space forms satisfy the local volume condition in Theorem 0.1. On the other hand, given any $\rho, \epsilon > 0$ and $H = \pm 1$ or 0 , there is a H -space form which contains a point x such that $\text{vol}(B_\rho(x)) < \epsilon$ i.e., $B_\rho(x)$ is collapsed.

In [CRX], we proposed the following quantitative version of Theorem 0.1.

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Conjecture 0.2. (Quantitative volume space form rigidity) Given $n, \rho, d > 0$ and $H = \pm 1$ or 0 , there exists a constant $\epsilon(n, \rho, d) > 0$ such that for any $0 < \epsilon < \epsilon(n, \rho, d)$, if a compact n -manifold M satisfies

$$\text{Ric}_M \geq (n-1)H, \quad d \geq \text{diam}(M), \quad \frac{\text{vol}(B_\rho(x^*))}{\text{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon, \quad \forall x \in M,$$

then M is diffeomorphic and $\Psi(\epsilon|n, \rho, d)$ -close in the Gromov-Hausdorff topology to a space form of constant curvature H , where $d = \pi$ or 1 when $H = 1$ or 0 respectively, where $\Psi(\epsilon|n, \rho, d) \rightarrow 0$ as $\epsilon \rightarrow 0$ while n, ρ and d are fixed.

Note that Conjecture 0.2 for $H = -1$ does not hold if one removes a bound on diameter (see [CRX]). On the other hand, for $H \neq -1$, M in Conjecture 0.2 may have arbitrarily small volume i.e., M is collapsed.

By the volume convergence ([Co2]), Conjecture 0.2 implies the following:

Conjecture 0.3. (Non-collapsing on Riemannian universal cover) Given $n, \rho, d > 0$, $H = \pm 1$ or 0 , there exist constants, $\epsilon(n, \rho, d), v(n, \rho, d) > 0$, such that if a compact n -manifold M satisfies

$$\text{Ric}_M \geq (n-1)H, \quad d \geq \text{diam}(M), \quad \frac{\text{vol}(B_\rho(x^*))}{\text{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon(n, \rho, d), \quad \forall x \in M,$$

then $\text{vol}(B_1(\tilde{x})) \geq v(n, \rho, d) > 0$, where \tilde{x} is a point in the Riemannian universal covering space.

In [CRX], among other things we proved that Conjecture 0.3 implies Conjecture 0.2 for $H \neq 1$, and for $H = 1$, Conjecture 0.2 holds when M is not collapsed. Precisely, the following theorem is a combination of Theorem A, B and C in [CRX] (corresponding to $H = 1, -1$ and 0).

Theorem 0.4. *Given $n, \rho, d, v > 0$ and $H = \pm 1$ or 0 , there exists a constant $\epsilon(n, \rho, d, v) > 0$ such that for any $0 < \epsilon < \epsilon(n, \rho, d, v)$, if a compact n -manifold M satisfies*

$$\text{Ric}_M \geq (n-1)H, \quad d \geq \text{diam}(M), \quad \text{vol}(B_1(z_0)) \geq v, \quad \frac{\text{vol}(B_\rho(x^*))}{\text{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon, \quad \forall x \in M,$$

then M is diffeomorphic and $\Psi(\epsilon|n, \rho, d, v)$ -close to a space form of constant curvature H , where $d = \pi$ or 1 when $H = 1$ or 0 , $z_0 \in M$ or $z_0 \in \tilde{M}$ when $H = 1$ or $H \neq 1$.

For $H = 1$, Theorem 0.4 generalizes the differential sphere theorem in [CC2] (cf. [Pe], [Co1], see Remark 0.7 in [CRX]), and for $H = -1$, Theorem 0.4 is equivalent to a quantitative version of the maximal volume entropy rigidity in [LW] (see Theorem D, Corollary 0.6 in [CRX]).

In the present paper, we will verify Conjecture 0.2 under an additional assumption: Ricci curvature is also bounded above (Theorem D). This regularity condition allows us to find a nearby metric of almost constant sectional curvature (Theorem B) by smoothing method ([DWY]) via renormalized Ricci flows in sense of [TW]. As an application we verify Conjecture 0.3 in this case (Theorem C).

We now begin to state the main results in this paper.

The first result says that under bounded Ricci curvature, the almost maximality of volume on local coverings measures how far the metric from being an H -Einstein metric (compare to Remark 0.5).

Theorem A. *Given $n, \rho, \Lambda > 0$, $H = \pm 1$ and 0 , there exists a constant, $\epsilon(n, \rho, \Lambda) > 0$, such that for $0 < \epsilon < \epsilon(n, \rho, \Lambda)$, if a compact Riemannian n -manifold (M, g) satisfies*

$$\Lambda \geq \text{Ric}(g) \geq (n-1)H, \quad \frac{\text{vol}(B_\rho(x^*))}{\text{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon, \quad \forall x \in M,$$

then g is almost Einstein in L^p -sense for any $p \geq 1$ i.e.,

$$\int_M |\text{Ric}(g) - (n-1)Hg|^p < \Psi(\epsilon|n, \rho, \Lambda, p).$$

The additional upper bound on Ricci curvature implies a uniform $C^{1,\alpha}$ -Harmonic radius on $B_{\frac{\rho}{2}}(x^*)$ (Lemma 1.3), and a local version of Theorem A on $B_{\frac{\rho}{2}}(x^*)$ (Lemma 1.4). By a packing argument via relative volume comparison, we obtain Theorem A.

Consider the Ricci flow on (M, g) ; following [DWY] we see that bounded Ricci curvature and a uniform $C^{1,\alpha}$ -Harmonic radius on $B_{\frac{\rho}{2}}(x^*)$ (independent of x) imply that the Ricci flow on M exists for a definite time (Theorem 1.5), and that the renormalized Ricci flow ([TW]) preserves the almost Einstein property in L^p -sense (Lemma 1.7). Using the two properties, we will prove the following strong smoothing result.

Theorem B. *(Smoothing to almost constant curvature) Given $n, \rho, \Lambda, \delta > 0$ and $H = \pm 1$ or 0 , there exists a constant, $\epsilon(n, \rho, \Lambda, \delta) > 0$, such that for any $0 < \epsilon < \epsilon(n, \rho, \Lambda, \delta)$, if a compact n -manifold (M, g) satisfies*

$$\Lambda \geq \text{Ric}(g) \geq (n-1)H, \quad \frac{\text{vol}(B_\rho(x^*))}{\text{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon, \quad \forall x \in M,$$

then M admits a metric g' such that $|g' - g| < \delta$ and for any $0 \leq k < \infty$,

$$|\text{Rm}(g')|_{C^k, M} \leq C(n, \rho, \Lambda, \delta, k), \quad H - \Psi(\delta, \epsilon|n, \rho, \Lambda) \leq \sec_{g'} \leq H + \Psi(\delta, \epsilon|n, \rho, \Lambda).$$

Using the existence of a nearby metric of almost constant sectional curvature, we are able to verify Conjecture 0.3 for the case bounded Ricci curvature.

Theorem C. *Given $n, \rho, d, \Lambda > 0$ and $H = \pm 1$ or 0 , there exist positive constants, $\epsilon(n, \rho, d, \Lambda), v(n, \rho, d, \Lambda) > 0$, such that if a compact n -manifold M satisfies*

$$\Lambda \geq \text{Ric}_M \geq (n-1)H, \quad d \geq \text{diam}(M), \quad \frac{\text{vol}(B_\rho(x^*))}{\text{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon(n, \rho, d, \Lambda), \quad \forall x \in M,$$

then \tilde{M} is not collapsed i.e., $\text{vol}(B_1(\tilde{p})) \geq v(n, \rho, d, \Lambda)$ for any $\tilde{p} \in \tilde{M}$, where $d = \pi$ or ∞ when $H = 1$ or -1 .

By Theorem C, we can apply Theorem 0.4 for $H \neq 1$ to verify Conjecture 0.2 for the case of bounded Ricci curvature. For $H = 1$ (bounded Ricci curvature), by the higher regularity of a nearby metric in Theorem B we are able to strengthen Theorem 3.5 in [CRX] to conclude Conjecture 0.2 in this case.

Theorem D. *Given $n, \rho, d, \Lambda > 0$ and $H = \pm 1$ or 0 , there exists a constant $\epsilon(n, \rho, d, \Lambda) > 0$ such that for any $0 < \epsilon < \epsilon(n, \rho, d, \Lambda)$, if a compact n -manifold M satisfies*

$$\Lambda \geq \text{Ric}_M \geq (n-1)H, \quad d \geq \text{diam}(M), \quad \frac{\text{vol}(B_\rho(x^*))}{\text{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon, \quad \forall x \in M,$$

then M is diffeomorphic and $\Psi(\epsilon|n, \rho, d, \Lambda)$ -close to a space form of constant curvature H , where $d = \pi$ or 1 when $H = 1$ or 0 respectively.

As mentioned in the above, there is a uniform lower bound on $C^{1,\alpha}$ -harmonic radius on $B_{\frac{\rho}{2}}(x^*)$ (see Lemma 1.3). Together with the above Theorem D and Theorem 2.1 in [CRX], we obtain the following $C^{1,\alpha}$ -compactness result.

Theorem E. *Given $n, \rho, d, \Lambda, v > 0$, there exist $\epsilon = \epsilon(n, \rho, d, \Lambda, v)$ such that the collection of compact n -manifolds satisfying*

$$\Lambda \geq \text{Ric}_M \geq (n-1)H, \quad d \geq \text{diam}(M), \quad \text{vol}(M) \geq v, \quad \frac{\text{vol}(B_\rho(x^*))}{\text{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon, \quad \forall x \in M$$

is compact in the $C^{1,\alpha}$ -topology, where the condition, “ $\text{vol}(M) \geq v$ ” can be removed when $H = -1$.

A few remarks are in order:

Remark 0.5. In the proof of Theorem A, we actually proved that g is almost Einstein on any $B_{\frac{\rho}{2}}(x)$, $x \in M$ (see (2.1.2)); compare to Problem 2.4. Roughly, one may interpret this as under bounded Ricci curvature, a ball with almost maximal ‘rewinding volume’ is an almost ‘Einstein ball’.

Remark 0.6. The existence of a nearby metric of almost constant curvature in Theorem B is crucial to our proof of Theorem C (and Theorem D). Indeed, we do not know, even assuming a higher regularity on the original metric, how to prove Theorem C without using a nearby metric of almost constant sectional curvature.

Remark 0.7. In Theorem C, no restriction on diameter for $H = -1$. For $H = 0$, the condition on bounded diameter cannot be removed. Here is a counterexample: for each i , let S_i^3 denote a round 3-sphere of radius i , and let g_i be a collapsed Berger’s metric such that $\text{vol}(B_1(p, g_i)) < i^{-1}$ and $i^{-5} < \sec(g_i) < 4i^{-2}$ (p.81, [Pet]). It is easy to see that $\frac{\text{vol}(B_1(p^*, g_i^*))}{\text{vol}(\underline{B}_1^0)} \rightarrow 1$, as $i \rightarrow \infty$.

Remark 0.8. Theorem D verifies Conjecture 0.2 (thus Conjecture 0.3) for the case that Ricci curvature is bounded above. Note that for $H = 0, 1$, M in Theorem D can be collapsed (comparing to Theorem 0.4 where M is not collapsed when $H = 1$).

Remark 0.9. Note that Theorem E and the $C^{1,\alpha}$ -compactness theorem in [An] may have only a ‘small’ overlap. This is because the local volume condition in Theorem E and the injectivity radius condition in [An] are somewhat ‘parallel’: a lower bound on injectivity radius may not imply the volume condition in Theorem E, and vice versa the volume conditions may not imply a lower bound on injectivity radius. (note that for $H = -1$, in Theorem E a priori M could be collapsed.)

The rest of the paper is organized as follows:

In Section 1, we will supply notions and basic properties that will be used through out the rest of the paper.

In Section 2, we will prove Theorems A-E. At the end, we will ask a few questions relating to the approach in this paper.

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1. PRELIMINARIES

The purpose of this section is to supply notions and basic properties that will be used through out the rest of the paper; we refer readers to [An], [CC1] and [DWY] for details.

a. Almost maximal volume ball is an almost space form ball.

Let N be a Riemannian $(n-1)$ -manifold, let $k : (a, b) \rightarrow \mathbb{R}$ be a smooth positive function and let $(a, b) \times_k N$ be the k -warped product whose Riemannian tensor is

$$g = dr^2 + k^2(r)g_N.$$

The Riemannian distance $|(r_1, x_1)(r_2, x_2)|$ ($x_1 \neq x_2$) equals to the infimum of the length

$$\int_0^l \sqrt{(c_1'(t))^2 + k^2(c_1(t))} dt$$

for any smooth curve $c(t) = (c_1(t), c_2(t))$ such that $c(0) = (r_1, x_1)$, $c(l) = (r_2, x_2)$ and $|c_2'| \equiv 1$, and $|(r_1, x)(r_2, x)| = |r_2 - r_1|$. Thus given a, b, k , there is a function (e.g., the law of cosine on space forms)

$$\rho_{a,b,k}(r_1, r_2, |x_1 x_2|) = |(r_1, x_2)(r_2, x_2)|.$$

Using the same formula for $|(r_1, x_2)(r_2, x_2)|$, one can extend the k -warped product $(a, b) \times_k Y$ to any metric space Y (not necessarily a length space); see [CC1].

The following theorem in [CC1] asserts that an almost volume annulus (see (1.1.1) below) is an almost metric annulus (see (1.1.2)).

Theorem 1.1 ([CC1]). *Let M be a Riemannian manifold, let r be a distance function to a compact subset in M , let $A_{a,b} = r^{-1}((a, b))$, let*

$$\mathcal{V}(u) = \inf \left\{ \frac{\text{vol}(B_u(q))}{\text{vol}(A_{a,b})} \mid \text{for all } q \in A_{a,b} \text{ with } B_u(q) \subset A_{a,b} \right\},$$

and let $0 < \alpha' < \alpha, \alpha - \alpha' > \xi > 0$. If

$$\text{Ric}_M \geq -(n-1) \frac{k''(a)}{k(a)} \quad (\text{on } r^{-1}(a)),$$

$$\Delta r \leq (n-1) \frac{k'(a)}{k(a)} \quad (\text{on } r^{-1}(a)),$$

$$(1.1.1) \quad \frac{\text{vol}(A_{a,b})}{\text{vol}(r^{-1}(a))} \geq (1-\epsilon) \frac{\int_a^b k^{n-1}(r) dr}{k^{n-1}(a)}.$$

Then there exists a length metric space Y , with at most $\#(a, b, k, \mathcal{V})$ components Y_i , satisfying

$$\text{diam}(Y_i) \leq D(a, b, k, \mathcal{V}),$$

such that the Gromov-Hausdorff distance,

$$(1.1.2) \quad d_{GH}(A_{a+\alpha, b-\alpha}, (a+\alpha, b-\alpha) \times_k Y) \leq \Psi(\epsilon | n, k, a, b, \alpha', \xi, \mathcal{V})$$

with respect to the two metrics $d^{\alpha', \alpha}$ and $\underline{d}^{\alpha', \alpha}$, where $d^{\alpha', \alpha}$ (resp. $\underline{d}^{\alpha', \alpha}$) denotes the restriction of the intrinsic metric of $A_{a+\alpha', b-\alpha'}$ on $A_{a+\alpha, b-\alpha}$ (resp. $(a+\alpha', b-\alpha') \times_k Y$) on $(a+\alpha, b-\alpha) \times_k Y$.

Let

$$\text{sn}_H(r) = \begin{cases} \frac{\sin \sqrt{H} r}{\sqrt{H}} & H > 0 \\ r & H = 0 \\ \frac{\sinh \sqrt{-H} r}{\sqrt{-H}} & H < 0 \end{cases}.$$

Applying Theorem 1.1 to $k = \text{sn}_H(r)$ with $r(x) = d(p, x) : M \rightarrow \mathbb{R}$, we conclude the following result that is used in the proof of Theorem A-E.

Theorem 1.2. *For $n, \rho, \epsilon > 0$, if a complete n -manifold M contains a point p satisfies*

$$\text{Ric}_M \geq (n-1)H, \quad \frac{\text{vol}(B_\rho(p))}{\text{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon,$$

then $d_{GH}(B_{\frac{\rho}{2}}(p), \underline{B}_{\frac{\rho}{2}}^H) < \Psi(\epsilon | n, \rho, H)$.

Note that $\frac{\text{vol}(B_\rho(p))}{\text{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon$ implies (1.1.1), as $a \rightarrow 0$. Since the almost maximal volume condition holds at all points near p (which contains regular points), by simple blow up argument one concludes that Y is isometric to S_1^{n-1} .

b. Almost maximal volume and $C^{1, \alpha}$ -harmonic radius estimate.

In this and the next subsections, we will always assume bounded Ricci curvature: $\Lambda \geq \text{Ric} \geq (n-1)H$, $H = \pm 1$ or 0 .

Let M be a complete n -manifold. For $p \in M$, $k \geq 0$, $0 < \alpha < 1$ and $Q \geq 1$, the $C^{k, \alpha}$ -harmonic radius at p with respect to Q is the largest radius $r_h(p)$ of the ball at p such that there are harmonic coordinates on $B_{r_h}(p)$ and $r^{k+\alpha} |g_{ij}|_{C^{k, \alpha}, B_{r_h}(p)} \leq Q$. The harmonic radius of a subset is the infimum of the harmonic radii of points in the subset.

Lemma 1.3. *For $n, \rho, \Lambda > 0$, $Q > 1$ and $0 < \alpha < 1$, there are constants, $\epsilon(n, \rho, \Lambda), r_h(n, \rho, \Lambda, Q, \alpha) > 0$, such that if a compact Riemannian n -manifold (M, g) satisfies*

$$\Lambda \geq \text{Ric}(g) \geq (n-1)H, \quad \frac{\text{vol}(B_\rho(x^*))}{\text{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon(n, \rho, \Lambda), \quad \forall x \in M,$$

the $C^{1,\alpha}$ -harmonic radius on $B_{\frac{\rho}{2}}(x^)$ with respect to Q is at least $r_h(n, \rho, \Lambda, Q, \alpha)$.*

Proof. We argue by contradiction, and the proof follows the same argument as in the proof Main Lemma 2.2 in [An]; where the almost maximal volume condition is replaced with a lower bound on injectivity radius which is to guarantee that any blow up limit is \mathbb{R}^n . We claim that a contradicting sequence, $\frac{\text{vol}(B_\rho(x_i^*))}{\text{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon_i \rightarrow 1$, also satisfies that any blow up limit is \mathbb{R}^n . Hence, the same proof in [An] goes through here to derive a contradiction.

To see the claim, for any $y_i^* \in B_{\frac{\rho}{2}}(x_i^*)$, $R > 0$ and $r_i \rightarrow \infty$, by Bishop-Gromov relative volume comparison and the volume convergence in [Co2] we derive

$$\frac{\text{vol}(B_R(y_i^*, r_i^2 g_i^*))}{\text{vol}(\underline{B}_R^{r_i^{-2}H})} = \frac{\text{vol}(B_{r_i^{-1}R}(y_i^*))}{\text{vol}(\underline{B}_{r_i^{-1}R}^H)} \geq \frac{\text{vol}(B_{\frac{\rho}{8}}(y_i^*))}{\text{vol}(\underline{B}_{\frac{\rho}{8}}^H)} \geq 1 - \Psi(\epsilon_i | n, \rho).$$

Since $r_i^{-2}H \rightarrow 0$, by Theorem 1.2 we conclude that

$$d_{GH}(B_R(y_i^*, r_i^2 g_i^*), \underline{B}_R^0) \rightarrow 0.$$

Since R is arbitrarily chosen, the desired claim follows. \square

As an application of Lemma 1.3, we will prove a non-collapsed local version of Theorem A.

Lemma 1.4. *Given $n, \rho, \Lambda > 0$ and $H = \pm 1$ and 0, there is $\epsilon(n, \rho, \Lambda) > 0$ such that for $0 < \epsilon < \epsilon(n, \rho, \Lambda)$, if a complete n -manifold (M, g, x) satisfies*

$$\Lambda \geq \text{Ric}(g) \geq (n-1)H, \quad \frac{\text{vol}(B_\rho(x))}{\text{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon,$$

then for all $p \geq 1$,

$$\int_{B_{\frac{\rho}{2}}(x)} |\text{Ric}(g) - (n-1)Hg|^p \leq \Psi(\epsilon | n, \rho, \Lambda, p).$$

Proof. Arguing by contradiction, assume a contradicting sequence, (M_i, g_i, x_i) , satisfying

$$\Lambda \geq \text{Ric}_{M_i} \geq (n-1)H, \quad \frac{\text{vol}(B_\rho(x_i))}{\text{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon_i \rightarrow 1,$$

but $\int_{B_{\frac{\rho}{2}}(x_i)} |\text{Ric}(g_i) - (n-1)Hg_i|^{p_0} \geq \delta_0 > 0$, for some $p_0 \geq 1$.

By Theorem 1.2, we may assume that $B_{\frac{\rho}{2}}(x_i) \xrightarrow{GH} \underline{B}_{\frac{\rho}{2}}^H$. By [CC2], we may assume that for i large, $B_{\frac{\rho}{2}}(x_i)$ are diffeomorphic to $\underline{B}_{\frac{\rho}{2}}^H$. From the expression of Ricci curvature in a harmonic coordinate, a bound on Ricci curvature implies that $g_i \rightarrow g_H$ in $L^{2,p}$ -norm for all $p \geq 1$. Consequently, $h_i = \text{Ric}(g_i) - (n-1)Hg_i \rightarrow h = \text{Ric}(g) - (n-1)Hg \equiv 0$ on $\underline{B}_{\frac{\rho}{2}}^H$ in L^p -norm, a contradiction. \square

c. Almost maximal volume and Ricci flows.

The main reference for this subsection is [DWY].

Let (M, g) be a compact Riemannian manifold. The Ricci flow was introduced by Hamilton as the solution of the following parabolic PDE,

$$\frac{\partial}{\partial t} g(t) = -2 \operatorname{Ric}(g(t)), \quad g(0) = g.$$

The solution always exists for a short time $t > 0$, and if the maximal flow time $T_{\max} < \infty$, then $\max |\operatorname{Rm}(g(t))| \rightarrow +\infty$ as $t \rightarrow T_{\max}$ ([Ha]).

A basic property of Ricci flow is that it improves the regularity of the initial metric ([Sh1,2]). However, the regularity depends on the flow time. For our purpose, a uniform definite flow time is important. We have

Theorem 1.5. *For $n, \rho, \Lambda > 0$ and $H = \pm 1$ or 0 , there are positive constants, $\epsilon(n, \rho, \Lambda), T(n, \rho, \Lambda)$, such that if a compact Riemannian n -manifold (M, g) satisfies*

$$\Lambda \geq \operatorname{Ric}(g) \geq (n-1)H, \quad \frac{\operatorname{vol}(B_\rho(x^*))}{\operatorname{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon(n, \rho, \Lambda), \quad \forall x \in M,$$

then the Ricci flow,

$$\frac{\partial}{\partial t} g(t) = -2 \operatorname{Ric}(g(t)), \quad g(0) = g,$$

exists for $t \in [0, T(n, \rho, \Lambda)]$ and

$$|g(t) - g| < 4t, \quad |\operatorname{Rm}(g(t))|_{C^k} \leq C, \quad \Lambda + ct^{\frac{1}{2}} \geq \operatorname{Ric}(g(t)) \geq (n-1)H - ct^{\frac{1}{2}},$$

where $C = C(n, \rho, \Lambda, k, t)$ and $c = c(n, \rho, \Lambda)$.

Note that Theorem 1.5 is similar to Theorem 1.1 in [DWY], where the volume condition on local covering is replaced by a positive lower bound on conjugate radius. Note that the condition on conjugate radius is solely used to show a $L^{2,p}$ -harmonic radius lower bound on a local covering space for all $p \geq 1$ (see Remark 1 in [DWY]), which is required to apply the Moser's weak Maximum principle (Theorem 2.1 in [DWY]). Because a lower bound on the $L^{2,p}$ -harmonic radius follows from Lemma 1.3 and bounded Ricci curvature condition, the same proof in [DWY] will give a proof of Theorem 1.5 with the obvious modification (cf. [Sh1,2]).

Let (M, g) be as in Theorem 1.5. Inspired by [DWY] we will show that if g is almost H -Einstein in L^p -sense, then the renormalized Ricci flow solution $g(t)$ in (1.6.1) below is again almost H -Einstein in L^p -sense (Lemma 1.7).

Consider the renormalized Ricci flow in the sense of [TW]:

$$(1.6.1) \quad \frac{\partial}{\partial t} g = -\operatorname{Ric}(g) + (n-1)Hg,$$

and let

$$\bar{g}(s) = \begin{cases} \sqrt{1 - 4(n-1)Hs} \cdot g\left(\frac{\ln(1-4(n-1)Hs)}{-2(n-1)H}\right), & H = \pm 1 \\ g(2s), & H = 0. \end{cases}$$

Then $\bar{g}(s)$ satisfies $\bar{g}(0) = g(0)$ and

$$(1.6.2) \quad \frac{\partial}{\partial s} \bar{g} = -2 \operatorname{Ric}(\bar{g}(s)).$$

Let $g^*(t)$ (resp. $\bar{g}^*(s)$) be the lifting of $g(t)$ (resp. $\bar{g}(s)$) on $B_\rho(x^*)$. Then

$$|\bar{R}_{ijkl}^*(s)| = \begin{cases} \frac{1}{\sqrt{1-4(n-1)Hs}} \cdot R_{ijkl}^* \left(\frac{\ln(1-4(n-1)Hs)}{-2(n-1)H} \right), & H = \pm 1 \\ R_{ijkl}^*(2s), & H = 0. \end{cases}$$

Let

$$(1.6.3) \quad h_{ij}^* = R_{ij}^* - (n-1)Hg_{ij}^*.$$

Then

$$\frac{\partial}{\partial t} h_{ij}^* = \frac{1}{2} \Delta h_{ij}^* + R_{pijq}^* h_{pq}^* - h_{ip}^* h_{pj}^*.$$

To get rid of the $\frac{1}{2}$ -factor, we make a change of variable $t = 2t'$ (for simple notation, switch by $t' = t$). Then the above implies

$$(1.6.4) \quad \frac{\partial}{\partial t} |h^*| \leq \Delta |h^*| + 2|Rm^*| |h^*|.$$

By applying the Moser's weak maximum principle, we conclude the following:

Lemma 1.7. *Let the assumptions be in Theorem 1.5, and let $h_{ij}^*(t)$ be defined in the above, and*

$$\bar{T}(n, \rho, \Lambda) = \begin{cases} \frac{\ln(1-4(n-1)HT(n, \rho, \Lambda))}{-4(n-1)H}, & H = \pm 1 \\ T(n, \rho, \Lambda), & H = 0. \end{cases}$$

Then for $t \in (0, \bar{T}(n, \rho, \Lambda)]$,

$$\max_{x \in M} |h^*(t)|_{p, B_{\frac{\rho}{4}}(x^*, g)} \leq \max_{x \in M} |h^*(0)|_{p, B_{\frac{\rho}{2}}(x^*, g)} \cdot \frac{1}{1 - c(n, \rho, \Lambda)t}.$$

Proof. Given Lemma 1.3, by (1.6.4) the rest proof is an imitation of the proof of Lemma 3.3 in [DWY]. \square

2. PROOF OF THEOREMS A-E

Proof of Theorem A.

Because Ricci curvature is bounded in absolute value, it suffices to prove Theorem A for $p = 1$. For any $x \in M$, by Lemma 1.4 we have

$$(2.1.1) \quad \int_{B_{\frac{\rho}{2}}(x^*)} |\operatorname{Ric}(g^*) - (n-1)Hg^*| \leq \Psi(\epsilon|n, \rho, \Lambda).$$

We claim that for $\Psi_1(\epsilon|n, \rho, \Lambda) = \Psi(\epsilon|n, \rho, \Lambda) \cdot \frac{\text{vol}(B_{\frac{\rho}{8}}^H)}{\text{vol}(B_{\frac{\rho}{16}}^H)}$, the following holds:

$$(2.1.2) \quad \int_{B_{\frac{\rho}{8}}(x)} |\text{Ric}(g) - (n-1)Hg| \leq \Psi_1(\epsilon|n, \rho, \Lambda).$$

Let $A = \{x_i\}$ denote an $\frac{\rho}{8}$ -net on M . Then $B_{\frac{\rho}{16}}(x_i) \cap B_{\frac{\rho}{16}}(x_j) = \emptyset$ ($i \neq j$) and $M \subseteq \bigcup_{x_i \in A} B_{\frac{\rho}{8}}(x_i)$. Assuming (2.1.2), we derive

$$\begin{aligned} \int_M |\text{Ric}(g) - (n-1)Hg| &\leq \frac{1}{\text{vol}(M)} \sum_{x_i \in A} \int_{B_{\frac{\rho}{8}}(x_i)} |\text{Ric}(g) - (n-1)Hg| \\ &= \frac{1}{\text{vol}(M)} \sum_{x_i} \text{vol}(B_{\frac{\rho}{8}}(x_i)) \int_{B_{\frac{\rho}{8}}(x_i)} |\text{Ric}(g) - (n-1)Hg| \\ &\leq \frac{1}{\text{vol}(M)} \sum_{x_i \in A} \text{vol}(B_{\frac{\rho}{8}}(x_i)) \Psi_1(\epsilon|n, \rho, \Lambda) \\ &\leq \frac{1}{\text{vol}(M)} \sum_{x_i \in A} \text{vol}(B_{\frac{\rho}{16}}(x_i)) \cdot \frac{\text{vol}(B_{\frac{\rho}{8}}^H)}{\text{vol}(B_{\frac{\rho}{16}}^H)} \Psi_1(\epsilon|n, \rho, \Lambda) \\ &\leq \Psi_2(\epsilon|n, \rho, \Lambda). \end{aligned}$$

We now verify (2.1.2). Let D denote the Dirichlet fundamental domain at $x^* \in \widetilde{B_\rho(x)}$, and let $\Gamma(\frac{\rho}{4}) = \{\gamma \in \pi_1(B_\rho(x^*)), |x^*\gamma(x^*)| \leq \frac{\rho}{4}\}$. Then

$$B_{\frac{\rho}{8}}(x^*) \subset \bigcup_{\gamma \in \Gamma(\frac{\rho}{4})} \gamma(B_{\frac{\rho}{8}}(x^*) \cap D) \subset B_{\frac{\rho}{2}}(x^*),$$

We claim that there is a $\gamma \in \Gamma(\frac{\rho}{4})$ such that

$$\int_{\gamma(B_{\frac{\rho}{8}}(x^*) \cap D)} |\text{Ric}(g^*) - (n-1)Hg^*| \leq \Psi_1(\epsilon|n, \rho, \Lambda),$$

i.e.,

$$\int_{B_{\frac{\rho}{8}}(x)} |\text{Ric}(g) - (n-1)Hg| \leq \Psi_1(\epsilon|n, \rho, \Lambda).$$

If the claim fails i.e., for all $\gamma \in \Gamma(\frac{\rho}{4})$,

$$\int_{\gamma(B_{\frac{\rho}{8}}(x^*) \cap D)} |\text{Ric}(g^*) - (n-1)Hg^*| > \Psi_1(\epsilon|n, \rho, \Lambda),$$

then

$$\begin{aligned} &\int_{B_{\frac{\rho}{2}}(x^*)} |\text{Ric}(g^*) - (n-1)Hg^*| \\ &\geq \sum_{\gamma \in \Gamma(\frac{\rho}{4})} \frac{\text{vol}(\gamma(B_{\frac{\rho}{8}}(x^*) \cap D))}{\text{vol}(B_{\frac{\rho}{2}}(x^*))} \int_{\gamma(B_{\frac{\rho}{8}}(x^*) \cap D)} |\text{Ric}(g^*) - (n-1)Hg^*| \\ &> \frac{\Psi_1(\epsilon|n, \rho, \Lambda)}{\text{vol}(B_{\frac{\rho}{2}}(x^*))} \sum_{\gamma \in \Gamma(\frac{\rho}{4})} \text{vol}(\gamma(B_{\frac{\rho}{8}}(x^*) \cap D)) \\ &\geq \frac{\Psi_1(\epsilon|n, \rho, \Lambda) \text{vol}(B_{\frac{\rho}{8}}(x^*))}{\text{vol}(B_{\frac{\rho}{2}}(x^*))} \\ &\geq \Psi(\epsilon|n, \rho, \Lambda), \end{aligned}$$

a contradiction to (2.1.1). \square

Proof of Theorem B.

Arguing by contradiction, assume a contradicting sequence, $(M_i, g_i) \xrightarrow{GH} X$, such that

$$\Lambda \geq \text{Ric}(g_i) \geq (n-1)H, \quad \frac{\text{vol}(B_\rho(x_i^*))}{\text{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon_i \rightarrow 0, \quad \forall x_i \in M_i,$$

and M_i admits no nearby metric to g_i with almost constant sectional curvature H .

Fixing a small $\delta \in (0, T(n, \rho, \Lambda)]$ (Theorem 1.5), let $g_i(\delta)$ denote the renormalized Ricci flow in (1.6.1). By Theorem 1.5, for any $x_i \in M_i$, passing to a subsequence we may assume that the lifting metric $g_i^*(\delta)$ on $B_\rho(x_i^*)$ satisfies

$$B_{\frac{\rho}{2}}(x_i^*, g_i^*(\delta)) \xrightarrow{C^k} B_{\frac{\rho}{2}}(x_\delta^*, g_\infty^*(\delta)), \quad h_i(g_i^*(\delta)) \xrightarrow{C^k} h(g_\infty^*(\delta)),$$

where h_i is defined in (1.6.3), and the C^k -convergence can be seen from the Cheeger-Gromov convergence theorem. Consequently, $g_\infty^*(\delta)$ is a smooth metric and $h(g_\infty^*(\delta))$ is a smooth tensor on $B_{\frac{\rho}{2}}(x_\delta^*, g_\infty^*(\delta))$. By Lemma 1.4 and Lemma 1.7, for any $x_i \in M_i$,

$$|h(g_i^*(\delta))|_{p, B_{\frac{\rho}{4}}(x_i^*)} \rightarrow 0.$$

Consequently, $h(g_\infty^*(\delta))|_{B_{\frac{\rho}{4}}(x_\delta^*, g_\infty^*(\delta))} \equiv 0$ i.e., $g_\infty^*(\delta)|_{B_{\frac{\rho}{4}}(x_\delta^*, g_\infty^*(\delta))}$ is H -Einstein.

Clearly, $B_{\frac{\rho}{4}}(x_\delta^*, g_\infty^*(\delta)) \xrightarrow{GH} \underline{B}_{\frac{\rho}{4}}^H$ as $\delta \rightarrow 0$. Since $g_\infty^*(\delta)$ is H -Einstein for all δ , $B_{\frac{\rho}{4}}(x_\delta^*, g_\infty^*(\delta)) \xrightarrow{C^k} \underline{B}_{\frac{\rho}{4}}^H$, for any $k \geq 1$ ([CC2]). Consequently, for δ_0 sufficiently small, $g_\infty^*(\delta_0)$ has almost constant sectional curvature H . Since $B_{\frac{\rho}{4}}(x_i^*, g_i^*(\delta_0)) \xrightarrow{C^k} B_{\frac{\rho}{4}}(x_{\delta_0}^*, g_\infty^*(\delta_0))$, for i large, $g_i^*(\delta_0)$ has almost constant curvature H . Since x_i is arbitrarily chosen, we conclude that $g_i(\delta_0)$ has almost constant sectional curvature H , a contradiction. \square

Proof of Theorem C.

Fixing a small $\delta > 0$, by Theorem B we may assume a nearby metric $g(\delta)$ such that

$$|g - g(\delta)| < \delta, \quad H - \delta \leq \sec_{g(\delta)} \leq H + \delta.$$

Case 1. Assume $H = -1$. For any $\tilde{p} \in \tilde{M}$, the exponential map, $\exp_{\tilde{p}}^{\tilde{g}(\delta)} : T_{\tilde{p}}\tilde{M} \rightarrow \tilde{M}$, is a diffeomorphism such that its differential has a bounded norm on $B_1(0)$ depending on n . Consequently, $\text{vol}(B_1(\tilde{p}, \tilde{g}(\delta)))$ has a positive lower bound depending only on n, ρ and d . Since $|\tilde{g} - \tilde{g}(\delta)| < \delta$, we conclude the desired result.

Case 2. Assume $H = 0$. By Splitting theorem of Cheeger-Gromoll, $\tilde{M} = \mathbb{R}^k \times N$, where N is a simply connected $(n-k)$ -manifold of non-negative Ricci curvature. We claim that N is a point. Note that $\text{diam}(N) \leq c(n, d)$ (see the proof of Theorem C in [CRX] where we normalize $d = 1$). We may assume $\delta^{-\frac{1}{2}} > 4 \text{diam}(N)$. Note that since $\sec_{g(\delta)} < \delta$, $\exp_{\tilde{p}}^{\tilde{g}(\delta)} : B_{\frac{1}{\sqrt{\delta}}}(0) \rightarrow B_{\frac{1}{\sqrt{\delta}}}(\tilde{p}, \tilde{g}(\delta))$ is a local diffeomorphism. Note that $B_{\frac{1}{2\sqrt{\delta}}}(\tilde{p})$ can be deformed to $0 \times N$ ($\tilde{p} = (0, x)$) and thus $B_{\frac{1}{2\sqrt{\delta}}}(\tilde{p})$ is simply connected. Consequently, the lifting of $B_{\frac{1}{2\sqrt{\delta}}}(\tilde{p})$ via $\exp_{\tilde{p}}^{\tilde{g}(\delta)}$ is contained in

the segment domain (i.e. each $\tilde{x} \in B_{\frac{1}{2\sqrt{\delta}}}(\tilde{p})$ is connecting to \tilde{p} by a unique minimal geodesic; if c_1 and c_2 are two distinct minimal geodesics, then $c_1 * c_2^{-1}$ is a loop at \tilde{p} , and so is the lifting of $c_1 * c_2^{-1}$ a loop at 0. Note that with respect to the pullback metric on $T_{\tilde{p}}\tilde{M}$, we obtain two geodesics from 0 to some v ; a contradiction). Therefore, $B_{\frac{1}{2\sqrt{\delta}}}(\tilde{p})$ is contractible in \tilde{M} , a contradiction.

Case 3. Assume $H = 1$. The classical 1/4-pinned injectivity radius estimate implies that the pullback metric $\tilde{g}(\delta)$ on \tilde{M} has injectivity radius $> \frac{\pi}{2}$, and thus $\text{vol}(B_1(\tilde{p}))$ has a positive lower bound depending on n . By now the desired result follows. \square

In the rest of the paper, we will freely use properties of equivariant Gromov-Hausdorff convergence; see b. of Section 1 in [CRX] for details.

Lemma 2.2. *Let M_i be a sequence of compact n -manifolds satisfying*

$$\text{Ric}_{M_i} \geq (n-1), \quad |\text{Rm}|_{C^1, M_i} \leq C, \quad \text{vol}(\tilde{M}_i) \geq v > 0,$$

and the commutative diagram,

$$\begin{array}{ccc} (\tilde{M}_i, \Gamma_i) & \xrightarrow{GH} & (\tilde{M}_\infty, G) \\ \downarrow \pi_i & & \downarrow \pi \\ M_i & \xrightarrow{GH} & X, \end{array}$$

Then for i large,

(2.2.1) *There is injective homomorphism and ϵ_i -GHA ($\epsilon_i \rightarrow 0$), $\phi_i : \Gamma_i \rightarrow G$, such that $\phi_i(\Gamma_i)$ acts freely on \tilde{M}_∞ .*

(2.2.2) *There is a Γ_i -conjugate diffeomorphism, $\tilde{f}_i : (\tilde{M}_i, \Gamma_i) \rightarrow (\tilde{M}_\infty, \phi_i(\Gamma_i))$, which is also an ϵ_i -GHA.*

Proof. Lemma 2.2 is essentially Theorem 3.5 in [CRX] where condition, “ $|\text{Rm}|_{C^1, M_i} \leq C$ ”, is replaced with “ $\frac{\text{vol}(B_\rho(\tilde{x}_i))}{\text{vol}(B_\rho^1)} \geq 1 - \epsilon_i \rightarrow 1$ ”, and the proof of Theorem 3.5 proves Lemma 2.2 with the following modifications: the regularity condition in Lemma 2.2 implies the following:

(i) There is a uniform lower bound on the injectivity radius of \tilde{M}_i , and thus \tilde{M}_∞ is a Riemannian manifold, and for any $r_i \rightarrow \infty$, passing to a subsequence $(\tilde{M}_i, \tilde{p}_i, r_i^2 \tilde{g}_i)$ converges to \mathbb{R}^n ; which guarantees (2.2.1).

(ii) \tilde{f} in (2.2.2) is a diffeomorphism, instead of a homotopy equivalence in Theorem 3.5; see the discussion following Theorem 3.5. \square

Proof of Theorem D.

By Theorem C, we may assume $\text{vol}(B_1(\tilde{p})) \geq v$. For $H \neq 1$, by Theorem 0.4 we obtain the desired conclusion (indeed, the case $H = 0$ has been already proved in the proof of Theorem C).

For $H = 1$, because $\text{vol}(M)$ can be very small, Theorem 0.4 cannot be applied here. Arguing by contradiction, assume a contradicting sequence, (M_i, g_i) , such that g_i satisfies the conditions of Theorem D for $\epsilon_i \rightarrow 0$ but none of M_i is diffeomorphic to a spherical space form.

For each i , let $g_i(\delta)$ be as in Theorem B, such that for all $1 \leq k < \infty$,

$$|\text{Rm}(g_i(\delta))|_{C^k} \leq C(n, \rho, \Lambda, \delta, k), \quad 1 - \delta \leq \sec_{g_i(\delta)} \leq 1 + \delta.$$

Passing to a subsequence we may assume the following commutative diagram:

$$\begin{array}{ccc} (\tilde{M}_i, \tilde{g}_i(\delta), \Gamma_i) & \xrightarrow{GH} & (\tilde{M}_\infty(\delta), \tilde{g}_\infty(\delta), G(\delta)) \\ \downarrow \pi_i & & \downarrow \pi \\ (M_i, g_i(\delta)) & \xrightarrow{GH} & (X, d_\infty(\delta)), \end{array}$$

where Γ_i denotes the deck transformations. Since \tilde{M}_i is not collapsed (Theorem C), by Lemma 2.2 there is a Γ_i -conjugate diffeomorphism, $\tilde{f}_i(\delta) : (\tilde{M}_i, \tilde{g}_i(\delta), \Gamma_i) \rightarrow (\tilde{M}_\infty(\delta), \tilde{g}_\infty(\delta), \phi_i(\delta)(\Gamma_i))$. From the proof of Theorem B, we see that $(\tilde{M}_\infty(\delta), \tilde{g}_\infty(\delta))$ is 1-Einstein. It is clear that $(\tilde{M}_\infty(\delta), \tilde{g}_\infty(\delta), G(\delta)) \xrightarrow{GH} (S_1^n, \underline{g}_1, G)$, as $\delta \rightarrow 0$. Consequently, for all $k < \infty$, $(\tilde{M}_\infty(\delta), \tilde{g}_\infty(\delta), G(\delta)) \xrightarrow{C^k} (S_1^n, \underline{g}_1, G)$ ([CC2]).

For each δ , we may choose i large such that $d_{GH}(\phi_i(\delta)(\Gamma_i), G(\delta)) < \delta_i \rightarrow 0$ i.e., $(M_\infty(\delta), \tilde{g}_\infty(\delta), \phi_i(\delta)(\Gamma_i)) \xrightarrow{GH} (S_1^n, \underline{g}_1, G)$. We then apply Lemma 2.2 again to conclude that for a fixed small δ , there is $\phi_i(\delta)(\Gamma_i)$ -conjugate diffeomorphism, $\tilde{f}_\infty(\delta) : (\tilde{M}_\infty(\delta), \phi_i(\delta)(\Gamma_i)) \rightarrow (S_1^n, \psi_i(\delta) \circ \phi_i(\delta)(\Gamma_i))$. Then $\tilde{f}_\infty(\delta) \circ \tilde{f}_i(\delta) : (\tilde{M}_i, \Gamma_i) \rightarrow (S_1^n, \psi_i(\delta) \circ \phi_i(\delta)(\Gamma_i))$ is Γ_i -conjugate diffeomorphism, and thus M_i is diffeomorphic to a spherical space form, $S_1^n / (\psi_i(\delta) \circ \phi_i(\delta)(\Gamma_i))$, a contradiction. \square

Remark 2.3. Given Theorem B, the conclusion of Theorem D for $H = 0$ and $H = 1$ can also be seen from the work [Gr] and [BS] respectively.

Proof of Theorem E.

It suffices to show that for any $Q \geq 1$ and $0 < \alpha < 1$, there is a constant $r_h = r_h(n, \rho, d, \Lambda, v, \alpha, Q) > 0$ such that M has $C^{1,\alpha}$ -harmonic radius with respect to Q bounded below by r_h ; because $\Lambda \geq \text{Ric}_M \geq (n-1)H$.

Arguing by contradiction, assume for some $Q_0 \geq 1$ and $0 < \alpha_0 < 1$, there is a contradicting sequence, M_i , satisfying

$$\Lambda \geq \text{Ric}_{M_i} \geq (n-1)H, \quad d \geq \text{diam}(M_i), \quad \frac{\text{vol}(B_\rho(x_i^*))}{\text{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon_i \rightarrow 1, \quad \forall x_i \in M_i,$$

and $p_i \in M_i$ such that the C^{1,α_0} -harmonic radius $r_h(p_i) \rightarrow 0$. Passing to a subsequence, we may assume the following commutative diagram:

$$\begin{array}{ccc} (\widetilde{B_\rho(p_i)}, p_i^*, K_i) & \xrightarrow{GH} & (X^*, p^*, K) \\ \downarrow \pi_i & & \downarrow \pi \\ (B_\rho(p_i), p_i) & \xrightarrow{GH} & (B_\rho(p), p), \end{array}$$

where K_i denotes the fundamental group of $B_\rho(p_i)$. Since $\frac{\text{vol}(B_\rho(p_i^*))}{\text{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon_i \rightarrow 1$, by Theorem 1.2 we see that $B_\rho(p^*)$ is local isometric to a H -space form. If $H \neq -1$, K_i is discrete because $\text{vol}(M_i) \geq v$. We claim that K is discrete when $H = -1$.

Hence, in any case we are able to apply Theorem 2.1 in [CRX] to conclude that K acts freely on X^* . We may assume that any element in K_i moves any x_i^* in $B_{\frac{\rho}{2}}(p_i^*)$ at least δ -distance, where δ depends on (X^*, K) . By Lemma 1.3, we may assume that $r_h(p_i^*) \geq r_h(n, \rho, \Lambda, \alpha_0, Q_0) > 0$, and thus $2r_h(p_i) \geq \min\{\delta, r_h(p_i^*)\} > 0$, a contradiction.

To see that K is discrete, note that by Theorem D we conclude that M_i is $\Psi(\epsilon|n, \rho, d, \Lambda)$ close to a hyperbolic manifold \mathbb{H}^n/Γ_i . By Margulis-Heintze lemma ([He]), \mathbb{H}^n/Γ_i is not collapsed, and by the volume convergence in [Co2] we then conclude that M_i is not collapsed (so $B_\rho(x_i)$ is not collapsed), and thus K is discrete. \square

We will conclude this paper with the following questions related to the present approach to Conjecture 0.3:

Problem 2.4. Does Theorem A hold without an upper bound on Ricci curvature? Indeed, it seems that even it is not known whether the scalar curvature is almost constant in L^p -sense.

Problem 2.5. (Ricci flow time) For $n, \rho > 0$, and $H = \pm 1$ or 0 , are there constants, $\epsilon(n, \rho) > 0, T(n, \rho) > 0$, such that for any $0 < \epsilon < \epsilon(n, \rho)$, if a compact n -manifold (M, g) satisfies

$$\text{Ric}(g) \geq (n-1)H, \quad \frac{\text{vol}(B_\rho(x^*))}{\text{vol}(\underline{B}_\rho^H)} \geq 1 - \epsilon, \quad \forall x \in M,$$

then the Ricci flow from g exists for $t \in [0, T(n, \rho)]$?

Problem 2.6. (Flows preserving almost Einstein) Let (M, g) be a compact n -manifold of $\text{Ric}_M \geq (n-1)H$ and

$$\int_{B_{\frac{\rho}{2}}(p^*)} |\text{Ric}(g^*) - (n-1)Hg^*| < \epsilon.$$

Let $g(t)$ be a renormalized Ricci flow of g (see (1.6.1)). Is $\int_{B_{\frac{\rho}{2}}(p^*, g^*(t))} |\text{Ric}(g^*(t)) - (n-1)Hg^*(t)| < \Psi(\epsilon|n, \rho, t)$.

Note that if there are affirmative answers to Problem 2.4-2.6, then the approach in this paper can be extended toward a proof of Conjecture 0.2.

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